A note on the FW 'mean-position' operator (Foldy and Wouthuysen)

1982 J. Phys. A: Math. Gen. 15 L259

(http://iopscience.iop.org/0305-4470/15/6/002)

J. Phys. A: Math. Gen. **15** (1982) L259–L260. Printed in Great Britain

LETTER TO THE EDITOR

A note on the FW 'mean-position' operator

J R Ellis and G Siopsis

School of Mathematical and Physical Sciences, University of Sussex, Falmer, Brighton BN1 9QH, England

Received 22 February 1982

Abstract. The frequently quoted expression for the 'mean-position' operator of Foldy and Wouthuysen is shown to be incorrect. A revised expression is given.

In a recent paper by one of us (Ellis 1981) it was pointed out that the formula for the 'mean-position' operator of Foldy and Wouthuysen (1950, formula (23)) which is also quoted in at least two texts (Schweber 1961, Margenau and Murphy 1964) is incorrect. In this letter we give a revised expression which satisfies the correct equation of motion for this operator. (Contrary to the statement made in their original article, the expression given by Foldy and Wouthuysen does not satisfy this equation.) The expression has been obtained independently by both of us, and has been derived by using the same lengthy but routine method indicated in Foldy and Wouthuysen's original paper.

The 'mean-position' operator $\boldsymbol{X}$ in the Dirac representation is the operator whose operator representative in the FW representation is $\boldsymbol{x}$. The transformation from $\boldsymbol{x}$ to $\boldsymbol{X}$ is

$$\boldsymbol{X} = U^{-1}\boldsymbol{x}U = \boldsymbol{x} + U^{-1}[\boldsymbol{x}, U]$$

where U is unitary, $U = \exp(\mathrm{i}S)$ and S is the Hermitian operator

$$S = -\frac{\mathrm{i}}{2m}\beta\boldsymbol{\alpha}\cdot\boldsymbol{p}\frac{\tan^{-1}(p/m)}{p/m}$$

(taking units in which $\hbar = c = 1$). Consequently,

$$U = \frac{E_p + \beta\boldsymbol{\alpha}\cdot\boldsymbol{p} + m}{[2E_p(E_p + m)]^{1/2}}$$

where E_p is the operator $(m^2 + p^2)^{1/2}$ (see, for example, Messiah 1962). Further, $[\boldsymbol{x}, E_p] = \mathrm{i}\boldsymbol{p}/E_p$. By induction, $[\boldsymbol{x}, E_p^n] = nE_p^{n-1}(\mathrm{i}\boldsymbol{p}/E_p)$; and for a function f of the operator E_p, $[\boldsymbol{x}, f(E_p)] = f'(E_p)(\mathrm{i}\boldsymbol{p}/E_p)$. For the 'mean-position' operator, we find

$$\boldsymbol{X} = \boldsymbol{x} + \frac{\mathrm{i}\beta\boldsymbol{\alpha}}{2E_p} - \frac{\mathrm{i}\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})\boldsymbol{p} + (\boldsymbol{\sigma}\wedge\boldsymbol{p})E_p}{2E_p^2(E_p + m)} \tag{1}$$

and this satisfies the desired equation $\mathrm{d}\boldsymbol{X}/\mathrm{d}t = -\mathrm{i}[\boldsymbol{X}, H] = (\boldsymbol{p}/E_p^2)H$, where H is the

Dirac Hamiltonian. The difference between (1) and the operator given by Foldy and Wouthuysen (1950, formula (23)) is

$$\mathrm{i}(E_p - p)\beta(\boldsymbol{\alpha} \cdot \boldsymbol{p})\boldsymbol{p}/2E_p^2 p(E_p + m)$$

and since this has *non-zero* time derivative the FW operator cannot satisfy the correct equation of motion.

References

Ellis J R 1981 *J. Phys. A: Math. Gen.* **14** 2917

Foldy L L and Wouthuysen S A 1950 *Phys. Rev.* **78** 29

Margenau H and Murphy G M 1964 *The Mathematics of Physics and Chemistry* vol 2 (Princeton, NJ: Van Nostrand) p 537

Messiah A 1962 *Quantum Mechanics* vol 2 (Amsterdam: North-Holland) p 942

Schweber S S 1961 *An Introduction to Relativistic Quantum Field Theory* (New York: Harper and Row) p 94